MONODROMY OPERATORS FOR HIGHER RANK

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ABSTRACT. We find an explicit form of the basic monodromy operator for the case of the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$. The expressions for the quantum Casimir elements of the quantum groups $\mathrm{U}_q(\mathfrak{sl}_3))$ and $\mathrm{U}_q(\mathfrak{gl}_3)$ are obtained as a byproduct.

To the memory of Yuri Stroganov.

1. INTRODUCTION

The most productive, although not comprehensive, approach to the theory of quantum integrable systems is based on the concept of a quantum group introduced by Drinfeld and Jimbo [13, 15]. In this approach, all the objects describing the model and related to its integrability originate from the universal R-matrix. For the first time, it was clearly realised by Bazhanov, Lukyanov and Zamolodchikov [2, 3, 4], see also [1, 8, 9]. The method was used to obtain an explicit form of R-matrices for certain representations of the quantum groups $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ [19, 20, 24, 11, 10, 6], $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ [24, 11, 10, 6] and $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3, \mu))$ [19, 7], where μ is the standard diagram automorphism of the Lie algebra $\mathfrak{sl}_3$ of order 2. An example of a quantum supergroup was considered in the paper [5]. It appears that one can also find the form of monodromy operators, transfer matrices, L-operators, and Q-operators [5, 6, 8].

The universal R-matrix is an element of the tensor product of two copies of the quantum group under consideration. A monodromy operator for a discrete quantum integrable system is constructed by a choice of two finite-dimensional representation of the quantum group. For the case of the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_n))$ the usual way to construct finite-dimensional representations is to use the homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_n))$ to $\mathrm{U}_q(\mathfrak{gl}_n)$ proposed by Jimbo [16]. It is convenient for applications to use this homomorphism for the first factor of the tensor product and some finite dimensional representation for the second factor. Here the monodromy operator is a matrix with entries in the quantum group $\mathrm{U}_q(\mathfrak{gl}_n)$. For the case of the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_2))$ such monodromy operator was obtained in the paper [8]. In the present paper we consider the case of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$. It is worth to note that the general form of the monodromy operator in question up to a factor belonging to the centre of $\mathrm{U}_q(\mathfrak{gl}_n)$ was found by Jimbo [16]. Our main goal is to find this factor. As a byproduct we obtain the expressions for the quantum Casimir elements of the quantum group $\mathrm{U}_q(\mathfrak{gl}_3)$.

Below $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers. Depending on the context, the symbol '1' means the number one, the unit of an algebra or the unit matrix. We use the notation

$$\kappa_q = q - q^{-1},$$

so that the quantum deformation of a number $\nu \in \mathbb{C}$ is

$$[\nu]_q = \frac{q^\nu - q^{-\nu}}{q - q^{-1}} = \kappa_q^{-1}(q^\nu - q^{-\nu}).$$

When necessary we identify a linear operator with its matrix with respect to a basis.

2. Quantum group $\mathrm{U}_q(\mathfrak{sl}_3)$

Depending on the sense of q, there are at least three definitions of a quantum group. According to the first definition, $q = \exp \hbar$, where $\hbar$ is an indeterminate, according to the second one, q is indeterminate, and according to the third one, $q = \exp \hbar$, where $\hbar$ is a complex number. In the first case a quantum group is a $\mathbb{C}[[\hbar]]$-algebra, in the second case a $\mathbb{C}(q)$-algebra, and in the third case it is just a complex algebra. Usually one uses monodromy operators to construct transfer operators with the help of some trace operation. It seems that to this end it is convenient to use the third definition of a quantum group. Therefore, we define the quantum group as a $\mathbb{C}$-algebra, see, for example, the books [17, 14].

Denote by $\mathfrak{h}$ the standard Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3$ and by H_i, $i = 1, 2$, the standard Cartan generators. The root system of $\mathfrak{sl}_3$ relative to $\mathfrak{h}$ is generated by the simple roots $\alpha_i \in \mathfrak{h}^*$, $i = 1, 2$, given by the relations

$$\alpha_j(H_i) = a_{ij}, \tag{2.1}$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.2}$$

is the Cartan matrix of $\mathfrak{sl}_3$.

Let $\hbar$ be a complex number and $q = \exp \hbar$. The quantum group $\mathrm{U}_q(\mathfrak{sl}_3)$ is a unital associative $\mathbb{C}$-algebra generated by the elements E_i, F_i, $i = 1, 2$, and q^X, $X \in \mathfrak{h}$, with the relations[1]

$$q^0 = 1, \qquad q^{X_1} q^{X_2} = q^{X_1 + X_2}, \tag{2.3}$$

$$q^X E_i q^{-X} = q^{\alpha_i(X)} E_i, \qquad q^X F_i q^{-X} = q^{-\alpha_i(X)} F_i, \tag{2.4}$$

$$[E_i, F_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \tag{2.5}$$

satisfied for any i and j, and the Serre relations

$$E_i^2 E_j - [2]_q E_i E_j E_i + E_j E_i^2 = 0, \qquad F_i^2 F_j - [2]_q F_i F_j F_i + F_j F_i^2 = 0 \tag{2.6}$$

satisfied for any distinct i and j. Note that q^X is just a convenient notation. There are no elements of $\mathrm{U}_q(\mathfrak{sl}_3)$ corresponding to the elements of $\mathfrak{h}$. Below for any $X \in gothh$ and $\nu \in \mathbb{C}$ we use the notation

$$q^{X + \nu} = q^\nu q^X.$$

Looking at (2.4) one can say that the generators e_i and f_i are related to the roots α_i and $-\alpha_i$ respectively. We define the elements related to the roots $\alpha_1 + \alpha_2$ and $-(\alpha_1 + \alpha_2)$ as

$$E_3 = E_1 E_2 - q^{-1} E_2 E_1, \qquad F_3 = F_2 F_1 - q F_1 F_2. \tag{2.7}$$

With respect to the properly defined coproduct, counit and antipode the quantum group $\mathrm{U}_q(\mathfrak{sl}_3)$ is a Hopf algebra.

There is a useful set of automorphisms of $\mathrm{U}_q(\mathfrak{sl}_3)$ defined as

$$E_i \to \nu_i E_i \, q^{\sum_{j=1}^2 \nu_{ij} H_j}, \qquad F_i \to \nu_i^{-1} q^{-\sum_{j=1}^2 \nu_{ij} H_j} F_i, \tag{2.8}$$

$$q^X \to q^X, \tag{2.9}$$

[1] Here and below we assume that $q^2 \neq 1$.

where the complex numbers ν_{ij} satisfy the relation

$$-\nu_{11} + 2\nu_{12} = 2\nu_{21} - \nu_{22},$$

and ν_i are arbitrary nonzero complex numbers.

3. Quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$

3.1. **Definition**

We start with the quantum group $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$. Remind that the Cartan subalgebra of $\widetilde{\mathcal{L}}(\mathfrak{sl}_3)$ is

$$\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c,$$

where $\mathfrak{h} = \mathbb{C}H$ is the standard Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3$ and c is the central element [18]. Define the Cartan elements

$$h_0 = c - H_1 - H_2, \qquad h_1 = H_1, \qquad h_2 = H_2,$$

so that one has

$$\widetilde{\mathfrak{h}} = \mathbb{C}h_0 \oplus \mathbb{C}h_1 \oplus \mathbb{C}h_2.$$

The simple positive roots $\alpha_i \in \widetilde{\mathfrak{h}}^*$, $i = 0, 1, 2$ are given by the equality

$$\alpha_j(h_i) = \tilde{a}_{ij},$$

where

$$(\tilde{a}_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

is the Cartan matrix of the Lie algebra $\widetilde{\mathcal{L}}(\mathfrak{sl}_3)$.

Let as before $\hbar$ be a complex number and $q = \exp \hbar$. The quantum group $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ is a unital associative $\mathbb{C}$-algebra generated by the elements e_i, f_i, $i = 0, 1, 2$, and q^x, $x \in \widetilde{\mathfrak{h}}$, with the relations

$$q^0 = 1, \qquad q^{x_1} q^{x_2} = q^{x_1 + x_2}, \tag{3.1}$$

$$q^x e_i q^{-x} = q^{\alpha_i(x)} e_i, \qquad q^x f_i q^{-x} = q^{-\alpha_i(x)} f_i, \tag{3.2}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \tag{3.3}$$

satisfied for all i and j, and the Serre relations

$$e_i^2 e_j - [2]_q e_i e_j e_i + e_j e_i^2 = 0, \qquad f_i^2 f_j - [2]_q f_i f_j f_i + f_j f_i^2 = 0 \tag{3.4}$$

satisfied for all distinct i and j.

The quantum group $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ is a Hopf algebra with the comultiplication Δ defined by the relations

$$\Delta(q^x) = q^x \otimes q^x, \tag{3.5}$$

$$\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i, \tag{3.6}$$

and with the correspondingly defined counit and antipode.

The quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ can be defined as the quotient algebra of $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ by the two-sided ideal generated by the elements of the form $q^{\nu c} - 1$, $\nu \in \mathbb{C}^\times$. In terms of generators and relations the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ is a $\mathbb{C}$-algebra generated by

the elements e_i, f_i, $i = 0, 1, 2$, and q^x, $x \in \widetilde{\mathfrak{h}}$, with relations (3.1)–(3.4) and an additional relation

$$q^{\nu(h_0+h_1+h_2)} = q^{\nu c} = 1, \tag{3.7}$$

where $\nu \in \mathbb{C}^\times$. It is a Hopf algebra with the comultiplication defined by (3.5), (3.6) and with the correspondingly defined counit and antipode. One of the reasons to use the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ instead of $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ is that in the case of $\mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ we have no expression for the universal R-matrix.

3.2. **Universal *R*-matrix**

As any Hopf algebra the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ has another comultiplication called the opposite comultiplication. It is given by the equalities

$$\Delta^{\mathrm{op}}(q^x) = q^x \otimes q^x,$$

$$\Delta^{\mathrm{op}}(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \qquad \Delta^{\mathrm{op}}(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i.$$

When the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ is defined as a $\mathbb{C}[[\hbar]]$-algebra it is quasitriangular. It means that there exists an element $\mathcal{R} \in \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3)) \otimes \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\, \Delta(a)\, \mathcal{R}^{-1}$$

for all $a \in \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$, and[2]

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}.$$

The most important property of the universal R-matrix is the equality

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} \tag{3.8}$$

called the Yang–Baxter equation for the universal R-matrix.

The expression for the universal R-matrix of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ considered as a $\mathbb{C}[[\hbar]]$-algebra can be constructed using the procedure proposed by Khoroshkin and Tolstoy [23]. Note that here the universal R-matrix is an element of $\mathrm{U}_q(\mathfrak{b}_+) \otimes \mathrm{U}_q(\mathfrak{b}_-)$, where $\mathrm{U}_q(\mathfrak{b}_+)$ is the Borel subalgebra of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ generated by e_i, $i = 0, 1, 2$, and q^x, $x \in \widetilde{\mathfrak{h}}$, and $\mathrm{U}_q(\mathfrak{b}_-)$ is the Borel subalgebra of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ generated by f_i, $i = 0, 1, 2$, and q^x, $x \in \widetilde{\mathfrak{h}}$.

In fact, one can use the expression for the universal R-matrix from the paper [23] also for the case of a quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ defined as a $\mathbb{C}$-algebra having in mind that in this case a quantum group is quasitriangular only in some restricted sense. Namely, all the relations involving the universal R-matrix should be considered as valid only for the weight representations of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$, see in this respect the paper [22] and the discussion below.

A representation π of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ on the vector space V is a weight representation if

$$V = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} V_\lambda,$$

where

$$V_\lambda = \{v \in V \mid q^x v = q^{\lambda(x)} v \text{ for any } x \in \widetilde{\mathfrak{h}}\}.$$

Note that the element λ in the definition of the weight subspace V_λ is defined uniquely. Therefore, for a given $x \in \widetilde{\mathfrak{h}}$ one can define the operator acting on $v \in V_\lambda$ as the multiplication by $\lambda(x)$. It is natural to denote this operator by $\pi(x)$.

[2]For the explanation of the notations see, for example, the book [12] or the papers [6, 8].

Let π_1 and π_2 be weight representations of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ on the vector spaces V_1 and V_2 with the weight decompositions

$$V_1 = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} (V_1)_\lambda, \qquad V_2 = \bigoplus_{\lambda \in \widetilde{\mathfrak{h}}^*} (V_2)_\lambda.$$

In the tensor product $V_1 \otimes V_2$ the role of the universal R-matrix is played by the operator

$$\mathcal{R}_{\pi_1, \pi_2} = (\pi_1 \otimes \pi_2)(\overline{\mathcal{R}})\, \mathcal{K}_{\pi_1, \pi_2}. \tag{3.9}$$

Here $\overline{\mathcal{R}}$ is an element of $\mathrm{U}_q(\mathfrak{n}_+) \otimes \mathrm{U}_q(\mathfrak{n}_-)$, where $\mathrm{U}_q(\mathfrak{n}_+)$ and $\mathrm{U}_q(\mathfrak{n}_-)$ are the subalgebras of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ generated by e_i, $i = 0, 1, 2$, and f_i, $i = 0, 1, 2$, respectively. The operator $\mathcal{K}_{\pi_1, \pi_2}$ acts on a vector $v \in (V_1)_{\lambda_1} \otimes (V_2)_{\lambda_2}$ in accordance with the equality

$$\mathcal{K}_{\pi_1, \pi_2}\, v = q^{\sum_{i,j=1}^2 b_{ij}\, \lambda_1(h_i) \lambda_2(h_j)}\, v, \tag{3.10}$$

where

$$(b_{ij}) = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is the inverse matrix of the Cartan matrix (2.2) of the Lie algebra $\mathfrak{sl}_3$. It is clear that

$$\mathcal{K}_{\pi_1, \pi_2} = q^{\sum_{i,j=1}^2 b_{ij}\, \pi_1(h_i) \otimes \pi_2(h_j)},$$

and, slightly abusing notation, we write

$$\mathcal{K}_{\pi_1, \pi_2} = (\pi_1 \otimes \pi_2)\Big(q^{\sum_{i,j=1}^2 b_{ij}\, h_i \otimes h_j}\Big) = (\pi_1 \otimes \pi_2)(\mathcal{K}).$$

In the present work we meet a more general situation. We have a homomorphism φ of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ into some other algebra A and a finite dimensional weight representation π of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ on the vector space V. Let $\{e_a\}$ be a basis of V consisting of weight vectors, λ_a an element of $\widetilde{\mathfrak{h}}^*$ corresponding to the vector e_a, and P_a the projection on e_a. Then the role of the universal R-matrix is played by the element

$$\mathcal{R}_{\varphi, \pi} = (\varphi \otimes \pi)(\overline{\mathcal{R}}) \mathcal{K}_{\varphi, \pi}$$

of $A \otimes \operatorname{End}(V)$ with

$$\mathcal{K}_{\varphi, \pi} = \sum_a \varphi\Big(q^{\sum_{i=1}^2 (\sum_{j=1}^2 b_{ij}\, \lambda_a(h_j))\, h_i}\Big) \otimes P_a. \tag{3.11}$$

Again, slightly abusing notation, we write

$$\mathcal{K}_{\varphi, \pi} = (\varphi \otimes \pi)\Big(q^{\sum_{i,j=1}^2 b_{ij}\, h_i \otimes h_j}\Big) = (\varphi \otimes \pi)(\mathcal{K}).$$

It is clear that in the case where φ is a representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ this definition is consistent with the definition (3.10).

To describe the structure of $\overline{\mathcal{R}}$ we have to define root vectors corresponding to the roots of $\widetilde{\mathcal{L}}(\mathfrak{sl}_3)$. We say that $a \in \mathrm{U}_q(\widetilde{\mathcal{L}}(\mathfrak{sl}_3))$ is a root vector corresponding to a root γ of $\widetilde{\mathcal{L}}(\mathfrak{sl}_3)$ if

$$q^x\, a\, q^{-x} = q^{\gamma(x)} a$$

for all $x \in \widetilde{\mathfrak{h}}$. It is customary to denote

$$\delta = \alpha_0 + \alpha_1 + \alpha_2, \qquad \alpha = \alpha_1, \qquad \beta = \alpha_2.$$

Let $\triangle_+ = \{\alpha, \beta, \alpha+\beta\}$ be the system of positive roots of $\mathfrak{sl}_3$. Then the system of positive roots of $\widetilde{\mathcal{L}}(\mathfrak{sl}_3)$ is [18]

$$\widetilde{\triangle}_+ = \{\gamma + k\delta \mid \gamma \in \triangle_+,\ k \in \mathbb{N}_0\} \\ \cup \{k\delta \mid k \in \mathbb{N}\} \cup \{(\delta - \gamma) + k\delta \mid \gamma \in \triangle_+,\ k \in \mathbb{N}_0\}.$$

The full system of roots is the union of the systems of positive and negative roots $\widetilde{\triangle} = \widetilde{\triangle}_+ \cup (-\widetilde{\triangle}_+)$.

We denote the root vector corresponding to a positive root γ by e_γ, and the root vector corresponding to a negative root γ by $f_{-\gamma}$. The evident choice for the root vectors corresponding to the simple roots is

$$e_{\delta-\alpha-\beta} = e_0, \qquad e_\alpha = e_1, \qquad e_\beta = e_2$$

and for the simple negative roots is

$$f_{\delta-\alpha-\beta} = f_0, \qquad f_\alpha = f_1, \qquad f_\beta = f_2.$$

We define the root vectors corresponding to the roots $\pm(\alpha+\beta)$ as

$$e_{\alpha+\beta} = e_\alpha\, e_\beta - q^{-1} e_\beta\, e_\alpha, \qquad f_{\alpha+\beta} = f_\beta\, f_\alpha - q\, f_\alpha\, f_\beta.$$

and the root vectors corresponding to the roots $\pm(\delta-\gamma)$, $\gamma \in \triangle_+$, as

$$e_{\delta-\alpha} = e_\beta\, e_{\delta-\alpha-\beta} - q^{-1} e_{\delta-\alpha-\beta}\, e_\beta, \qquad e_{\delta-\beta} = e_\alpha\, e_{\delta-\alpha-\beta} - q^{-1} e_{\delta-\alpha-\beta}\, e_\alpha,$$
$$f_{\delta-\alpha} = f_{\delta-\alpha-\beta}\, f_\beta - q\, f_\beta\, f_{\delta-\alpha-\beta}, \qquad f_{\delta-\beta} = f_{\delta-\alpha-\beta}\, f_\alpha - q\, f_\alpha\, f_{\delta-\alpha-\beta}.$$

The root vectors corresponding to the roots $\pm\delta$ are indexed by the elements of $\triangle_+$ and defined by the relations[3]

$$e'_{\delta,\gamma} = e_\gamma e_{\delta-\gamma} - q^{-2} e_{\delta-\gamma} e_\gamma, \qquad f'_{\delta,\gamma} = f_{\delta-\gamma} f_\gamma - q^{-2} f_\gamma f_{\delta-\gamma}.$$

Now we define the root vectors corresponding to the remaining roots $\pm(\gamma + k\delta)$ and $\pm((\delta-\gamma)+k\delta)$ as follows

$$e_{\gamma+k\delta} = [2]_q^{-1}(e_{\gamma+(k-1)\delta}\, e'_{\delta,\gamma} - e'_{\delta,\gamma}\, e_{\gamma+(k-1)\delta}), \tag{3.12}$$
$$e_{(\delta-\gamma)+k\delta} = [2]_q^{-1}(e'_{\delta,\gamma}\, e_{(\delta-\gamma)+(k-1)\delta} - e_{(\delta-\gamma)+(k-1)\delta}\, e'_{\delta,\gamma}), \tag{3.13}$$
$$f_{\gamma+k\delta} = [2]_q^{-1}(f'_{\delta,\gamma}\, f_{\gamma+(k-1)\delta} - f_{\gamma+(k-1)\delta}\, f'_{\delta,\gamma}), \tag{3.14}$$
$$f_{(\delta-\gamma)+k\delta} = [2]_q^{-1}(f_{(\delta-\gamma)+(k-1)\delta}\, f'_{\delta,\gamma} - f'_{\delta,\gamma}\, f_{(\delta-\gamma)+(k-1)\delta}). \tag{3.15}$$

The last step is to define the root vectors corresponding to the roots $k\delta$. For $k > 0$ they are defined as

$$e'_{k\delta,\gamma} = e_{\gamma+(k-1)\delta}\, e_{\delta-\gamma} - q^{-2} e_{\gamma-\delta}\, e_{\gamma+(k-1)\delta}, \tag{3.16}$$

and for $k < 0$ as

$$f'_{k\delta,\gamma} = f_{\delta-\gamma}\, f_{\gamma+(k-1)\delta} - q^2 f_{\gamma+(k-1)\delta}\, f_{\gamma-\delta},$$

where $\gamma \in \triangle_+$. The second type of vectors corresponding to the roots $k\delta$ is defined by the equation

$$e_{\delta,\gamma}(\zeta) = \kappa_q^{-1} \log\bigl(1 + \kappa_q e'_{\delta,\gamma}(\zeta)\bigr), \qquad f_{\delta,\gamma}(\zeta) = -\kappa_q^{-1} \log\bigl(1 - \kappa_q f'_{\delta,\gamma}(\zeta)\bigr), \tag{3.17}$$

[3]The prime stands to distinguish two types of the root vectors, see the definition below.

where

$$e'_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} e'_{k\delta,\gamma}\,\zeta^k, \qquad e_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} e_{k\delta,\gamma}\,\zeta^k,$$

$$f'_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} f'_{k\delta,\gamma}\,\zeta^k, \qquad f_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} f_{k\delta,\gamma}\,\zeta^k.$$

It is useful to have in mind that all $e_{k\delta,\gamma}$ commute and all $f_{k\delta,\gamma}$ commute as well.

The next ingredient of the Khoroshkin–Tolstoy construction is a normal order of $\triangle_+$. We use the following one [10]

$$\alpha,\ \alpha+\beta,\ \alpha+\delta, \alpha+\beta+\delta,\ \alpha+2\delta,\ \ \alpha+\beta+2\delta,\ \ldots\ ,\ \beta,\ \beta+\delta,\ \beta+2\delta,\ \ldots\ ,$$
$$\delta,\ 2\delta,\ \ldots\ ,\ \ldots\ ,\ (\delta-\beta)+2\delta,\ (\delta-\beta)+\delta,\ \delta-\beta,\ \ldots\ ,$$
$$(\delta-\alpha)+2\delta,\ (\delta-\alpha-\beta)+2\delta,\ (\delta-\alpha)+\delta,\ (\delta-\alpha-\beta)+\delta,\ \delta-\alpha,\ \delta-\alpha-\beta.$$

After all, $\overline{\mathcal{R}}$ is constructed as the product of three factors

$$\overline{\mathcal{R}} = \mathcal{R}_{\prec\delta}\,\mathcal{R}_{\sim\delta}\,\mathcal{R}_{\succ\delta}. \tag{3.18}$$

The factor $\mathcal{R}_{\prec\delta}$ is the product over $\gamma \in \triangle_+$ and $k \in \mathbb{N}_0$ of the q-exponentials

$$\mathcal{R}_{\gamma+k\delta} = \exp_{q^{-2}}(\kappa_q\, e_{\gamma+k\delta} \otimes f_{\gamma+k\delta}).$$

The order of the factors in $\mathcal{R}_{\prec\delta}$ coincides with the chosen normal order of the roots $\gamma+k\delta$. For the second factor in (3.18) we have

$$\mathcal{R}_{\sim\delta} = \exp\biggl(\kappa_q \sum_{k=1}^{\infty}\sum_{i,j=1}^{2} u_{k,ij}\, e_{k\delta,\alpha_i} \otimes f_{k\delta,\alpha_j}\biggr), \tag{3.19}$$

where for each $k \in \mathbb{N}$ the quantities $u_{k,ij}$ are the entries of the matrix

$$u_k = \frac{k}{[k]_q}\frac{1}{q^{2k}+1+q^{-2k}}\begin{pmatrix} q^k+q^{-k} & (-1)^k \\ (-1)^k & q^k+q^{-k}\end{pmatrix}. \tag{3.20}$$

The last factor in (3.18) is the product over $\gamma \in \triangle_+$ and $k \in \mathbb{N}_0$ of the q-exponentials

$$\mathcal{R}_{(\delta-\gamma)+k\delta} = \exp_{q^{-2}}\bigl(\kappa_q\, e_{(\delta-\gamma)+k\delta} \otimes f_{(\gamma-\delta)+k\delta}\bigr).$$

The order of the factors in $\mathcal{R}_{\succ\delta}$ coincides with the chosen normal order of the roots $(\delta-\gamma)+k\delta$.

4. R-MATRIX AND MONODROMY OPERATORS

We construct objects related to integrability by choosing representations for the factors of the tensor product $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3)) \otimes \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ and then applying them to the universal R-matrix [8]. A spectral parameter is introduced by endowing $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ with a $\mathbb{Z}$-gradation. To this end we use the following procedure [8]. Given $\zeta \in \mathbb{C}^\times$, we define an automorphism Φ_ζ of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ by its action on the generators of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ as

$$\Phi_\zeta(q^x) = q^x, \qquad \Phi_\zeta(e_i) = \zeta^{s_i} e_i, \qquad \Phi_\zeta(f_i) = \zeta^{-s_i} f_i, \tag{4.1}$$

where s_i are arbitrary integers. The automorphisms Φ_ζ corresponds to the $\mathbb{Z}$-gradation with the grading subspaces

$$\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))_m = \{a \in \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3)) \mid \Phi_\zeta(a) = \zeta^m a\}.$$

Note that for any $\zeta \in \mathbb{C}^\times$ the universal R-matrix of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ satisfies the condition

$$(\Phi_\zeta \otimes \Phi_\zeta)(\mathcal{R}) = \mathcal{R}. \tag{4.2}$$

Below we use the notation

$$s_\delta = s_0 + s_1 + s_2, \qquad s_\alpha = s_1, \qquad s_\beta = s_2, \qquad s_{\alpha+\beta} = s_1 + s_2.$$

4.1. *R*-matrix

The first useful object is an R-operator, or the R-matrix associated with it. To define the corresponding representation we use the Jimbo's homomorphism

$$\varphi : \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3)) \to \mathrm{U}_q(\mathfrak{sl}_3)$$

defined by the relations

$$\varphi(q^{\nu h_0}) = q^{-\nu(H_1 + H_2)}, \qquad \varphi(q^{\nu h_1}) = q^{\nu H_1}, \qquad \varphi(q^{\nu h_2}) = q^{\nu H_2}, \tag{4.3}$$

$$\varphi(e_0) = F_3\, q^{-(H_1 - H_2)/3}, \qquad \varphi(e_1) = E_1, \qquad \varphi(e_2) = E_2, \tag{4.4}$$

$$\varphi(f_0) = E_3\, q^{(H_1 - H_2)/3}, \qquad \varphi(f_1) = F_1, \qquad \varphi(f_2) = F_2, \tag{4.5}$$

see the paper [16].[4] Note that this is not a homomorphism of Hopf algebras.

We denote by $\pi^{(m_1, m_2)}$ the finite dimensional representation of $\mathrm{U}_q(\mathfrak{sl}_3)$ with the highest weight (m_1, m_2), $m_1, m_2 \in \mathbb{N}_0$. It is a highest weight representation with the highest weight vector v_0 characterised by the equalities

$$q^{\nu H_1} v_0 = q^{\nu m_1} v_0, \qquad q^{\nu H_2} v_0 = q^{\nu m_2} v_0, \qquad E_1 v_0 = 0, \qquad E_2 v_0 = 0.$$

Using the Jimbo's homomorphism we define the finite dimensional representation

$$\varphi^{(m_1, m_2)} = \pi^{(m_1, m_2)} \circ \varphi$$

of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$, and the finite dimensional representation of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ depending on the spectral parameter

$$\varphi^{(m_1, m_2)}(\zeta) = \pi^{(m_1, m_2)} \circ \varphi \circ \Phi_\zeta.$$

The first fundamental representation $\pi^{(1,0)}$ of $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ can be realised on the space $\mathbb{C}^3$ as

$$\pi^{(1,0)}(q^{\nu H_1}) = q\, E_{11} + q^{-1} E_{22}, \qquad \pi^{(1,0)}(q^{\nu H_2}) = q\, E_{22} + q^{-1} E_{33}, \tag{4.6}$$

$$\pi^{(1,0)}(E_1) = E_{12}, \qquad \pi^{(1,0)}(E_2) = E_{23}, \tag{4.7}$$

$$\pi^{(1,0)}(F_1) = E_{21}, \qquad \pi^{(1,0)}(F_2) = E_{32}, \tag{4.8}$$

where $E_{ab} \in \operatorname{End}(\mathbb{C}^3)$, $a, b = 1, 2, 3$, are defined by their action on the vectors of the standard basis $\{e_a\}$ of $\mathbb{C}^3$:

$$E_{ab} e_c = \delta_{bc} e_a.$$

Using the well known relations

$$E_{ab} E_{cd} = \delta_{bc} E_{ad},$$

we see that

$$\pi^{(1,0)}(E_3) = E_{13}, \qquad \pi^{(1,0)}(F_3) = E_{31}. \tag{4.9}$$

The R-operator associated with the representation $\pi^{(1,0)}$ is defined as

$$R(\zeta_1 | \zeta_2) = \big(\varphi^{(1,0)}(\zeta_1) \otimes \varphi^{(1,0)}(\zeta_2)\big)(\mathcal{R}).$$

[4] In fact, Jimbo defines a homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ to $\mathrm{U}_q(\mathfrak{gl}_3)$, see in this respect section 5.

It follows from (4.2) that

$$R(\zeta_1\nu|\zeta_2\nu) = R(\zeta_1|\zeta_2),$$

therefore, one has

$$R(\zeta_1|\zeta_2) = R(\zeta_1\zeta_2^{-1}),$$

where $R(\zeta) = R(\zeta|1)$. One can write

$$R(\zeta) = \sum_{a,b,c,d=1}^{3} E_{ac} \otimes E_{bd}\, \boldsymbol{R}_{ab|cd}(\zeta) \tag{4.10}$$

and define the 9×9 matrix

$$\boldsymbol{R}(\zeta) = (\boldsymbol{R}_{ab|cd}(\zeta)).$$

It can be shown [10, 6] that

$$\boldsymbol{R}(\zeta) = \overline{\boldsymbol{R}}(\zeta)\boldsymbol{D}. \tag{4.11}$$

Here for the non-zero quantities of the matrix $\overline{\boldsymbol{R}}(\zeta) = (\overline{\boldsymbol{R}}_{ab|cd}(\zeta))$ we have

$$\overline{\boldsymbol{R}}_{11|11}(\zeta) = \overline{\boldsymbol{R}}_{22|22}(\zeta) = \overline{\boldsymbol{R}}_{33|33}(\zeta) = f(\zeta^{s_\delta})\,(1 - q^{-2}\zeta^{s_\delta}), \tag{4.12}$$

$$\begin{aligned}\overline{\boldsymbol{R}}_{12|12}(\zeta) &= \overline{\boldsymbol{R}}_{13|13}(\zeta) = \overline{\boldsymbol{R}}_{21|21}(\zeta)\\ &= \overline{\boldsymbol{R}}_{23|23}(\zeta) = \overline{\boldsymbol{R}}_{31|31}(\zeta) = \overline{\boldsymbol{R}}_{32|32}(\zeta) = f(\zeta^{s_\delta})\,(1 - \zeta^{s_\delta}),\end{aligned} \tag{4.13}$$

$$\overline{\boldsymbol{R}}_{12|21}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\alpha}\kappa_q, \qquad \overline{\boldsymbol{R}}_{13|31}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\alpha+s_\beta}\kappa_q, \tag{4.14}$$

$$\overline{\boldsymbol{R}}_{23|32}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\beta}\kappa_q, \qquad \overline{\boldsymbol{R}}_{21|12}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta-s_\alpha}\kappa_q, \tag{4.15}$$

$$\overline{\boldsymbol{R}}_{31|13}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta-s_\alpha-s_\beta}\kappa_q, \qquad \overline{\boldsymbol{R}}_{32|23}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta-s_\beta}\kappa_q, \tag{4.16}$$

where $f(\zeta)$ a transcendental function having the form

$$f(\zeta) = \exp(\lambda_3(q^2\zeta^{s_\delta}) + \lambda_3(\zeta^{s_\delta}) + \lambda_3(q^{-4}\zeta^{s_\delta}))$$

with

$$\lambda_3(\zeta) = \sum_{k=0}^{\infty} \frac{1}{q^{2k} + 1 + q^{-2k}} \frac{\zeta^k}{k} = \sum_{k=0}^{\infty} \frac{1}{[3]_{q^k}} \frac{\zeta^k}{k}. \tag{4.17}$$

The non-zero quantities of the matrix $\boldsymbol{D} = (\boldsymbol{D}_{ab|cd})$ are

$$\boldsymbol{D}_{11|11} = \boldsymbol{D}_{22|22} = \boldsymbol{D}_{33|33} = q^{2/3}, \tag{4.18}$$

$$\boldsymbol{D}_{12|12} = \boldsymbol{D}_{13|13} = \boldsymbol{D}_{21|21} = \boldsymbol{D}_{23|23} = \boldsymbol{D}_{31|31} = \boldsymbol{D}_{32|32} = q^{-1/3}. \tag{4.19}$$

In fact, to define an R-operator one can use any finite dimensional representation of $\mathrm{U}_q(\mathfrak{sl}_3)$. Different choices correspond to different quantum integrable models. In this work we restrict ourselves to the models defined by the first fundamental representation.

4.2. **Ansatz for the monodromy operator**

To construct monodromy operators one uses different representations for different factors of the tensor product $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3)) \otimes \mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$. We use for the first factor the representation $\varphi^{(m_1,m_2)}(\zeta)$, for the second factor the representation $\varphi^{(1,0)}(\eta)$, and denote

$$M^{(m_1,m_2)}(\zeta|\eta) = ((\varphi^{(m_1,m_2)}(\zeta) \otimes \varphi^{(1,0)}(\eta))(\mathcal{R}).$$

From the point of view of spin chains such an operator corresponds to a one-site chain. In general, one takes instead of the representation $\varphi^{(1,0)}(\eta)$ a tensor product of representations such as $\varphi^{(1,0)}(\eta_1) \otimes \cdots \otimes \varphi^{(1,0)}(\eta_N)$ [8].

In fact, it is convenient for applications to define the monodromy operator

$$M(\zeta|\eta) = ((\varphi(\zeta) \otimes \varphi^{(1,0)}(\eta))(\mathcal{R})$$

and use the relation

$$M^{(m_1,m_2)}(\zeta|\eta) = (\pi^{(m_1,m_2)} \otimes \mathrm{id})(M(\zeta|\eta)).$$

Using the equality (4.2), one can demonstrate that

$$M(\zeta\nu|\eta\nu) = M(\zeta|\eta)$$

for any $\nu \in \mathbb{C}^\times$. Therefore, one has

$$M(\zeta|\nu) = M(\zeta\eta^{-1}),$$

where $M(\zeta) = M(\zeta|1)$. The operator $M(\zeta)$ can be represented as

$$M(\zeta) = \sum_{a,b=1}^{3} \mathbb{M}_{ab}(\zeta) \otimes E_{ab}, \tag{4.20}$$

where $\mathbb{M}_{ab}(\zeta)$ are appropriate unique elements of $\mathrm{U}_q(\mathfrak{sl}_3)$. Introduce the matrix

$$\mathbb{M}(\zeta) = (\mathbb{M}_{ab}(\zeta)).$$

By definition, it is an element of $\mathrm{Mat}_3(\mathrm{U}_q(\mathfrak{sl}_3))$. Let $\boldsymbol{P} = (\boldsymbol{P}_{ab|cd})$ be the matrix defined by the equality

$$\boldsymbol{P}_{ab|cd} = \delta_{ad}\,\delta_{bc}.$$

The corresponding linear operator is the permutation of the factors of the tensor product $\mathbb{C}^3 \otimes \mathbb{C}^3$. It follows from the Yang–Baxter equation (3.8) that

$$\hat{\boldsymbol{R}}(\zeta_1\zeta_2^{-1})(\mathbb{M}(\zeta_1) \boxtimes \mathbb{M}(\zeta_2)) = (\mathbb{M}(\zeta_2) \boxtimes \mathbb{M}(\zeta_1))\hat{\boldsymbol{R}}(\zeta_1\zeta_2^{-1}), \tag{4.21}$$

where $\hat{\boldsymbol{R}}(\zeta) = \boldsymbol{P}\boldsymbol{R}(\zeta)$, and $\boxtimes$ is a natural generalisation of the Kronecker product to the case of matrices with noncommuting entries [8].

One can write

$$M^{(m_1,m_2)}(\zeta) = \sum_{a,b=1}^{3} \pi^{(m_1,m_2)}(\mathbb{M}^3_{ab=1}(\zeta)) \otimes E_{ab} = \sum_{a,b} \mathbb{M}^{(m_1,m_2)}_{ab}(\zeta) \otimes E_{ab},$$

and define the matrix

$$\mathbb{M}^{(m_1,m_2)}(\zeta) = (\mathbb{M}^{(m_1,m_2)}_{ab}(\zeta)).$$

It is clear that one has

$$R(\zeta) = M^{(1,0)}(\zeta) = (\pi^{(1,0)} \otimes \mathrm{id})(M(\zeta)).$$

Therefore,

$$\mathbb{M}^{(1,0)}(\zeta) = \mathbb{R}(\zeta), \tag{4.22}$$

where

$$\mathbb{R}(\zeta) = (\mathbb{R}_{ab}(\zeta)),$$

and the quantities $\mathbb{R}_{ab}(\zeta) \in \mathrm{Mat}_3(\mathbb{C})$ are defined by the equality

$$R(\zeta) = \sum_{a,b=1}^{3} \mathbb{R}_{ab}(\zeta) \otimes E_{ab} = \sum_{a,b=1}^{3} \Big(\sum_{c,d=1}^{3} \boldsymbol{R}_{ca|db}(\zeta) E_{cd} \Big) \otimes E_{ab},$$

see (4.10) for the definition of $\boldsymbol{R}_{ab|cd}(\zeta)$. In accordance with (4.11) we have

$$\mathbb{R}(\zeta) = \overline{\mathbb{R}}(\zeta)\mathbb{D}.$$

Here $\overline{\mathbb{R}}(\zeta) = (\overline{\mathbb{R}}_{ab}(\zeta))$ and $\mathbb{D} = (\mathbb{D}_{ab})$, where, as follows from (4.12)–(4.16), (4.6)–(4.8) and (4.9), the matrices $\overline{\mathbb{R}}_{ab}(\zeta) \in \mathrm{Mat}_3(\mathbb{C})$ can be represented as

$$\overline{\mathbb{R}}_{11}(\zeta) = f(\zeta^{s_\delta})\,\pi^{(1,0)}\big(1 - q^{-2/3}\zeta^{s_\delta} q^{-(4H_1+2H_2)/3}\big), \tag{4.23}$$

$$\overline{\mathbb{R}}_{22}(\zeta) = f(\zeta^{s_\delta})\,\pi^{(1,0)}\big(1 - q^{-2/3}\zeta^{s_\delta} q^{(2H_1-2H_2)/3}\big), \tag{4.24}$$

$$\overline{\mathbb{R}}_{33}(\zeta) = f(\zeta^{s_\delta})\,\pi^{(1,0)}\big(1 - q^{-2/3}\zeta^{s_\delta} q^{(2H_1+4H_2)/3}\big), \tag{4.25}$$

$$\overline{\mathbb{R}}_{12}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta - s_\alpha}\kappa_q\,\pi^{(1,0)}(F_1), \quad \overline{\mathbb{R}}_{13}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta - s_{\alpha+\beta}}\kappa_q \pi^{(1,0)}(F_3), \tag{4.26}$$

$$\overline{\mathbb{R}}_{23}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\delta - s_\beta}\kappa_q\,\pi^{(1,0)}(F_2), \quad \overline{\mathbb{R}}_{21}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\alpha}\kappa_q\,\pi^{(1,0)}(E_1), \tag{4.27}$$

$$\overline{\mathbb{R}}_{31}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_{\alpha+\beta}}\kappa_q\,\pi^{(1,0)}(E_3), \quad \overline{\mathbb{R}}_{32}(\zeta) = f(\zeta^{s_\delta})\,\zeta^{s_\beta}\kappa_q\,\pi^{(1,0)}(E_2), \tag{4.28}$$

while, as follows from (4.18), (4.19) and (4.6), for the matrices $\mathbb{D}_{ab} \in \mathrm{Mat}_3(\mathbb{C})$ we have

$$\mathbb{D}_{11} = \pi^{(1,0)}(q^{(2H_1+H_2)/3}), \qquad \mathbb{D}_{22} = \pi^{(1,0)}(q^{-(H_1-H_2)/3}), \tag{4.29}$$

$$\mathbb{D}_{33} = \pi^{(1,0)}(q^{-(H_1+2H_2)/3}). \tag{4.30}$$

The equalities (4.23)–(4.30) suggest to assume that

$$\mathbb{M}(\zeta) = \overline{\mathbb{M}}(\zeta)\mathbb{K}. \tag{4.31}$$

Here $\overline{\mathbb{M}}(\zeta)$ is the matrix of the form

$$\overline{\mathbb{M}}(\zeta) = \mathrm{e}^{\Lambda(\zeta^{s_\delta})} \begin{pmatrix} \overline{\mathbb{M}}'_{11}(\zeta^{s_\delta}) & \zeta^{s_\delta - s_\alpha}\overline{\mathbb{M}}'_{12} & \zeta^{s_\delta - s_{\alpha+\beta}}\overline{\mathbb{M}}'_{13} \\ \zeta^{s_\alpha}\overline{\mathbb{M}}'_{21} & \overline{\mathbb{M}}'_{22}(\zeta^{s_\delta}) & \zeta^{s_\delta - s_\beta}\overline{\mathbb{M}}'_{23} \\ \zeta^{s_{\alpha+\beta}}\overline{\mathbb{M}}'_{31} & \zeta^{s_\beta}\overline{\mathbb{M}}'_{32} & \overline{\mathbb{M}}'_{33}(\zeta^{s_\delta}), \end{pmatrix} \tag{4.32}$$

where

$$\overline{\mathbb{M}}'_{11}(\zeta) = 1 - q^{-2/3}\zeta\, q^{-(4H_1+2H_2)/3}, \qquad \overline{\mathbb{M}}'_{22}(\zeta) = 1 - q^{-2/3}\zeta\, q^{(2H_1-2H_2)/3}, \tag{4.33}$$

$$\overline{\mathbb{M}}'_{33}(\zeta) = 1 - q^{-2/3}\zeta\, q^{(2H_1+4H_2)/3}, \tag{4.34}$$

$$\overline{\mathbb{M}}'_{12} = c_1\kappa_q\, F_1\, q^{c_{11}H_1 + c_{12}H_2}, \qquad \overline{\mathbb{M}}'_{21} = d_1\kappa_q\, E_1\, q^{d_{11}H_1+d_{12}H_2}, \tag{4.35}$$

$$\overline{\mathbb{M}}'_{23} = c_2\kappa_q\, F_2\, q^{c_{21}H_1 + c_{22}H_2}, \qquad \overline{\mathbb{M}}'_{32} = d_2\kappa_q\, E_2\, q^{d_{21}H_1+d_{22}H_2}, \tag{4.36}$$

$$\overline{\mathbb{M}}'_{13} = c_3\kappa_q\, F_3\, q^{c_{31}H_1 + c_{32}H_2}, \qquad \overline{\mathbb{M}}'_{31} = d_3\kappa_q\; E_3\, q^{d_{31}H_1+d_{32}H_2}, \tag{4.37}$$

$\Lambda(\zeta)$ belongs to the centre of $\mathrm{U}_q(\mathfrak{sl}_3)$, and $\mathbb{K}$ is a constant diagonal matrix with the diagonal entries

$$\mathbb{K}_{11} = q^{(2H_1+H_2)/3}, \qquad \mathbb{K}_{22} = q^{-(H_1-H_2)/3}, \qquad \mathbb{K}_{33} = q^{-(H_1+2H_2)/3}. \tag{4.38}$$

Substituting the ansatz (4.31) into the equality (4.21), we see that, up to an automorphism of the form (2.8) and (2.9), it is satisfied if we put

$$\overline{\mathbb{M}}'_{12} = \kappa_q\, q^{1/3} F_1\, q^{-(H_1+2H_2)/3}, \qquad \overline{\mathbb{M}}'_{21} = \kappa_q\, E_1, \tag{4.39}$$

$$\overline{\mathbb{M}}'_{23} = \kappa_q\, q^{1/3} F_2\, q^{(2H_1+H_2)/3}, \qquad \overline{\mathbb{M}}'_{32} = \kappa_q\, E_2, \tag{4.40}$$

$$\overline{\mathbb{M}}'_{13} = \kappa_q\, q^{1/3} F_3\, q^{-(H_1-H_2)/3}, \qquad \overline{\mathbb{M}}'_{31} = \kappa_q\; E_3. \tag{4.41}$$

The quantity $\Lambda(\zeta)$ remains arbitrary. In the next section we prove that the equalities (4.38)–(4.41) really describe the monodromy operator obtained from the universal R-matrix using the mapping $\varphi(\zeta) \otimes \varphi^{(1,0)}$ and find the expression for $\Lambda(\zeta)$.

4.3. Sketch of the proof

The proof is rather cumbersome and quite technical, therefore we only describe the main steps and leave the details to the reader.

Let us first discuss the general structure of the monodromy operator $M(\zeta)$. We have

$$M(\zeta) = U(\zeta)V(\zeta)W(\zeta)K,$$

where

$$U(\zeta) = \big(\varphi(\zeta) \otimes \varphi^{(1,0)}\big)(\mathcal{R}_{\prec\delta}), \qquad V(\zeta) = \big(\varphi(\zeta) \otimes \varphi^{(1,0)}\big)(\mathcal{R}_{\sim\delta})$$
$$W(\zeta) = \big(\varphi(\zeta) \otimes \varphi^{(1,0)}\big)(\mathcal{R}_{\succ\delta}), \qquad K = \big(\varphi(\zeta) \otimes \varphi^{(1,0)}\big)(\mathcal{K}).$$

Using relations similar to (4.20), we define the matrices $\mathbb{U}(\zeta)$, $\mathbb{V}(\zeta)$, $\mathbb{W}(\zeta)$ and $\mathbb{K}$ with the entries in $\mathrm{U}_q(\mathfrak{sl}_3)$ corresponding to the operators $U(\zeta)$, $V(\zeta)$, $W(\zeta)$ and K respectively.

Using (3.11), (4.3), (4.6) and taking into account that the endomorphism E_{aa} is the projection on the vector e_a of the standard basis of $\mathbb{C}^3$, we see that (4.38) is the right expression for the nonzero entries of the matrix $\mathbb{K}$. Thus, it remains to demonstrate that under the appropriate choice of $\mathrm{e}^{\Lambda(\zeta)}$ the equality

$$\mathbb{U}(\zeta)\mathbb{V}(\zeta)\mathbb{W}(\zeta) = \overline{\mathbb{M}}(\zeta), \tag{4.42}$$

where $\overline{\mathbb{M}}(\zeta)$ is determined by the relations (4.32) and (4.39)–(4.41), is true.

Note that for any $\gamma \in \triangle_+$ one has

$$\varphi(\zeta)(e_{\gamma+k\delta}) = \zeta^{s_\gamma + k s_\delta}\varphi(e_{\gamma+k\delta}), \qquad \varphi(\zeta)(e_{(\delta-\gamma)+k\delta}) = \zeta^{(s_\delta - s_\gamma)+k s_\delta}\varphi(e_{(\delta-\gamma)+k\delta}).$$

Further, in the same way as in the paper [6], we obtain

$$\varphi^{(1,0)}(f_{\alpha+k\delta}) = (-1)^k q^{4k/3} E_{21}, \qquad \varphi^{(1,0)}(f_{(\delta-\alpha)+k\delta}) = (-1)^k q^{(4k+1)/3} E_{12},$$
$$\varphi^{(1,0)}(f_{\beta+k\delta}) = q^{7k/3} E_{32}, \qquad \varphi^{(1,0)}(f_{(\delta-\beta)+k\delta}) = -q^{(7k+4)/3} E_{23},$$
$$\varphi^{(1,0)}(f_{\alpha+\beta+k\delta}) = (-1)^k q^{4k/3} E_{31}, \qquad \varphi^{(1,0)}(f_{(\delta-\alpha-\beta)+k\delta}) = (-1)^k q^{(4k+1)/3} E_{13}.$$

For $\gamma \in \triangle_+$ denote

$$\mathbb{E}_\gamma(\zeta) = \sum_{k=0}^{\infty} \varphi(e_{\gamma+k\delta})\,\zeta^k, \qquad \mathbb{E}_{\delta-\gamma}(\zeta) = \sum_{k=0}^{\infty} \varphi(e_{(\delta-\gamma)+k\delta})\,\zeta^k.$$

Using the properties of the endomorphisms E_{ab}, we see that the matrices $\mathbb{U}(\zeta)$ and $\mathbb{W}(\zeta)$ has the form

$$\mathbb{U}(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ \zeta^{s_\alpha}\mathbb{U}'_{21}(\zeta^{s_\delta}) & 1 & 0 \\ \zeta^{s_{\alpha+\beta}}\mathbb{U}'_{31}(\zeta^{s_\delta}) & \zeta^{s_\beta}\mathbb{U}'_{32}(\zeta^{s_\delta}) & 1 \end{pmatrix},$$

$$\mathbb{W}(\zeta) = \begin{pmatrix} 1 & \zeta^{s_\delta - s_\alpha}\mathbb{W}'_{12}(\zeta^{s_\delta}) & \zeta^{s_\delta - s_{\alpha+\beta}}\mathbb{W}'_{13}(\zeta^{s_\delta}) \\ 0 & 1 & \zeta^{s_\delta - s_\beta}\mathbb{W}'_{23}(\zeta^{s_\delta}) \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\mathbb{U}'_{21}(\zeta) = \kappa_q\,\mathbb{E}_\alpha(-q^{4/3}\zeta), \qquad \mathbb{U}'_{31}(\zeta) = \kappa_q\,\mathbb{E}_{\alpha+\beta}(-q^{4/3}\zeta), \tag{4.43}$$
$$\mathbb{U}'_{32}(\zeta) = \kappa_q\,\mathbb{E}_\beta(q^{7/3}\zeta), \qquad \mathbb{W}'_{12}(\zeta) = \kappa_q\,q^{1/3}\,\mathbb{E}_{\delta-\alpha}(-q^{4/3}\zeta), \tag{4.44}$$

$$\mathbb{W}'_{13}(\zeta) = \kappa_q\, q^{1/3}\, \mathbb{E}_{\delta-\alpha-\beta}(-q^{4/3}\zeta), \qquad \mathbb{W}'_{23}(\zeta) = -\kappa_q\, q^{4/3}\, \mathbb{E}_{\delta-\beta}(q^{7/3}\zeta). \tag{4.45}$$

Further, it is easy to get convinced that

$$\varphi(\zeta)(e_{k\delta,\gamma}) = \zeta^{s_\delta}\varphi(e_{k\delta,\gamma})$$

for any $\gamma \in \triangle_+$. Similarly as in the paper [6] we obtain

$$\varphi^{(1,0)}(f_{k\delta,\alpha}) = -(-1)^k \frac{[k]_q}{k} q^{k/3}(E_{11} - q^{2k}E_{22}),$$

$$\varphi^{(1,0)}(f_{k\delta,\beta}) = -\frac{[k]_q}{k} q^{4k/3}(E_{22} - q^{2k}E_{33}).$$

Using these relations and the definition (3.19), we see that the matrix $\mathbb{V}(\zeta)$ is of the diagonal form:

$$\mathbb{V}(\zeta) = \begin{pmatrix} \mathbb{V}'_{11}(\zeta^{s_\delta}) & 0 & 0 \\ 0 & \mathbb{V}'_{22}(\zeta^{s_\delta}) & 0 \\ 0 & 0 & \mathbb{V}_{33}(\zeta^{s_\delta}) \end{pmatrix}.$$

while the equality (3.20) gives

$$\log \mathbb{V}'_{11}(\zeta) = -\kappa_q \sum_{k=1}^{\infty} \frac{\big((-1)^k(q^k+q^{-k})\varphi(e_{k\delta,\alpha}) + \varphi(e_{k\delta,\beta})\big)\, q^{k/3}\zeta^k}{q^{2k}+1+q^{-2k}}, \tag{4.46}$$

$$\log \mathbb{V}'_{22}(\zeta) = \kappa_q \sum_{k=1}^{\infty} \frac{\big((-1)^k q^{3k}\varphi(e_{k\delta,\alpha}) - \varphi(e_{k\delta,\beta})\big)\, q^{k/3}\zeta^k}{q^{2k}+1+q^{-2k}}, \tag{4.47}$$

$$\log \mathbb{V}'_{33}(\zeta) = \kappa_q \sum_{k=1}^{\infty} \frac{\big((-1)^k\varphi(e_{k\delta,\alpha}) + (q^k+q^{-k})\varphi(e_{k\delta,\beta})\big)\, q^{10k/3}\zeta^k}{q^{2k}+1+q^{-2k}}. \tag{4.48}$$

It follows from (4.46) and (4.47) that

$$\log \mathbb{V}'_{22}(\zeta) - \log \mathbb{V}'_{11}(\zeta) = \kappa_q \mathbb{E}_{\delta,\alpha}(-q^{4/3}\zeta^{s_\delta}). \tag{4.49}$$

Here and below for $\gamma \in \triangle_+$ we use the notation

$$\mathbb{E}_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} \varphi(e_{k\delta,\gamma})\, \zeta^k.$$

The definition (3.17) gives

$$1 + \kappa_q \mathbb{E}'_{\delta,\gamma}(\zeta) = \exp\big(\kappa_q \mathbb{E}_{\delta,\gamma}(\zeta)\big),$$

where

$$\mathbb{E}'_{\delta,\gamma}(\zeta) = \sum_{k=1}^{\infty} \varphi(e'_{k\delta,\gamma})\, \zeta^k.$$

Hence, it follows from (4.49) that

$$1 + \kappa_q \mathbb{E}'_{\delta,\alpha}(\zeta) = \mathbb{V}'^{-1}_{11}(-q^{-4/3}\zeta)\mathbb{V}'_{22}(-q^{-4/3}\zeta). \tag{4.50}$$

In the same way, using (4.47) and (4.48), we obtain

$$1 + \kappa_q \mathbb{E}'_{\delta,\beta}(\zeta) = \mathbb{V}'^{-1}_{22}(q^{-7/3}\zeta)\mathbb{V}'_{33}(q^{-7/3}\zeta). \tag{4.51}$$

In fact, using different pairs of relations from (4.46)–(4.48) we obtain different expressions for $\mathbb{E}'_{\delta,\gamma}(\zeta)$, $\gamma \in \triangle_+$. However, they are related one to another by the identity

$$\mathbb{V}'_{11}(q^2\zeta)\mathbb{V}'_{22}(\zeta)\mathbb{V}'_{33}(q^{-2}\zeta) = 1 \tag{4.52}$$

which can be obtained from (4.46)–(4.48).

Rewrite the equality (4.42) in the component form and resolve the obtained equalities with respect to the entries of the matrices $\mathbb{U}$, $\mathbb{V}$ and $\mathbb{W}$. We come to the system

$$\mathbb{U}'_{21}(\zeta) = \overline{\mathbb{M}}'_{21}\overline{\mathbb{M}}'^{-1}_{11}(\zeta), \qquad \mathbb{U}'_{31}(\zeta) = \overline{\mathbb{M}}'_{31}\overline{\mathbb{M}}'^{-1}_{11}(\zeta), \tag{4.53}$$

$$\mathbb{U}'_{32}(\zeta) = \overline{\mathbb{M}}''_{32}(\zeta)\overline{\mathbb{M}}''^{-1}_{22}(\zeta), \qquad \mathbb{W}'_{12}(\zeta) = \overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}'_{12}, \tag{4.54}$$

$$\mathbb{W}'_{13}(\zeta) = \overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}'_{13}, \qquad \mathbb{W}'_{23}(\zeta) = \overline{\mathbb{M}}''^{-1}_{22}(\zeta)\overline{\mathbb{M}}''_{23}(\zeta), \tag{4.55}$$

$$\mathbb{V}'_{11}(\zeta) = \mathrm{e}^{\Lambda(\zeta)}\overline{\mathbb{M}}'_{11}(\zeta), \qquad \mathbb{V}'_{22}(\zeta) = \mathrm{e}^{\Lambda(\zeta)}\overline{\mathbb{M}}''_{22}(\zeta), \tag{4.56}$$

$$\mathbb{V}'_{33}(\zeta) = \mathrm{e}^{\Lambda(\zeta)}\big(\overline{\mathbb{M}}''_{33}(\zeta) - \zeta\,\overline{\mathbb{M}}''_{32}(\zeta)\overline{\mathbb{M}}''^{-1}_{22}(\zeta)\overline{\mathbb{M}}''_{23}(\zeta)\big), \tag{4.57}$$

where

$$\overline{\mathbb{M}}''_{23}(\zeta) = \overline{\mathbb{M}}'_{23} - \overline{\mathbb{M}}'_{21}\overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}'_{13},$$
$$\overline{\mathbb{M}}''_{32}(\zeta) = \overline{\mathbb{M}}'_{32} - \zeta\,\overline{\mathbb{M}}'_{31}\overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}_{12},$$
$$\overline{\mathbb{M}}''_{22}(\zeta) = \overline{\mathbb{M}}'_{22}(\zeta) - \zeta\,\overline{\mathbb{M}}'_{21}\overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}'_{12},$$
$$\overline{\mathbb{M}}''_{33}(\zeta) = \overline{\mathbb{M}}'_{33}(\zeta) - \zeta\,\overline{\mathbb{M}}'_{31}\overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}'_{13}.$$

It follows from (4.43)–(4.45) that the equalities (4.53)–(4.55) are equivalent to the equalities

$$\mathbb{E}_{\alpha}(\zeta) = \kappa_q^{-1}\overline{\mathbb{M}}'_{21}\overline{\mathbb{M}}'^{-1}_{11}(-q^{-4/3}\zeta), \tag{4.58}$$

$$\mathbb{E}_{\beta}(\zeta) = \kappa_q^{-1}\overline{\mathbb{M}}''_{32}(q^{-7/3}\zeta)\overline{\mathbb{M}}''^{-1}_{22}(q^{-7/3}\zeta), \tag{4.59}$$

$$\mathbb{E}_{\alpha+\beta}(\zeta) = \kappa_q^{-1}\overline{\mathbb{M}}'_{31}\overline{\mathbb{M}}'^{-1}_{11}(-q^{-4/3}\zeta), \tag{4.60}$$

$$\mathbb{E}_{\delta-\alpha}(\zeta) = \kappa_q^{-1}\overline{\mathbb{M}}'^{-1}_{11}(-q^{-4/3}\zeta)\overline{\mathbb{M}}'_{12}, \tag{4.61}$$

$$\mathbb{E}_{\delta-\beta}(\zeta) = -\kappa_q^{-1}q^{-4/3}\overline{\mathbb{M}}''^{-1}_{22}(q^{-7/3}\zeta)\overline{\mathbb{M}}''_{23}(q^{-7/3}\zeta), \tag{4.62}$$

$$\mathbb{E}_{\delta-\alpha-\beta}(\zeta) = \kappa_q^{-1}q^{-1/3}\overline{\mathbb{M}}'^{-1}_{11}(-q^{-4/3}\zeta)\overline{\mathbb{M}}'_{13}. \tag{4.63}$$

The equalities (3.12) and (3.13) give

$$\mathbb{E}_{\gamma}(\zeta) - \varphi(e_{\gamma}) = [2]_q^{-1}\zeta\,[\mathbb{E}_{\gamma}(\zeta),\, \varphi(e'_{\delta,\gamma})],$$
$$\mathbb{E}_{\delta-\gamma}(\zeta) - \varphi(e_{\delta-\gamma}) = [2]_q^{-1}\zeta\,[\varphi(e'_{\delta,\gamma}),\, \mathbb{E}_{\delta-\gamma}(\zeta)].$$

These relations determine $\mathbb{E}_{\gamma}(\zeta)$ and $\mathbb{E}_{\delta-\gamma}(\zeta)$ uniquely. One can verify that the right hand sides of (4.58)–(4.63) satisfy them. Hence, the equalities (4.58)–(4.63) are true and, therefore, the equalities (4.53)–(4.54) are also true.

Consider now the first equality of (4.56). Represent $\Lambda(\zeta)$ as

$$\Lambda(\zeta) = \sum_{k=1}^{\infty} \frac{C_k}{q^{2k} + 1 + q^{-2k}}\,\frac{\zeta^k}{k} \tag{4.64}$$

cf. the definition (4.17). Taking into account (4.33) and (4.46), we see that the first equality of (4.56) is true if an only if

$$C_k = (q^{2k} + 1 + q^{-2k})\,q^{-2k/3}q^{-(4H_1+2H_2)k/3} \\ - \kappa_q\,q^{k/3}k\big((-1)^k(q^k + q^{-k})\varphi(e_{k\delta,\alpha}) + \varphi(e_{k\delta,\beta})\big).$$

One can show that

$$e_{k\delta,\gamma} = \sum_{\ell_1+2\ell_2+\cdots+k\ell_k=k} \frac{(-\kappa_q)^{\ell_1+\ell_2+\cdots+\ell_k-1}(\ell_1+\ell_2+\cdots+\ell_k-1)!}{\ell_1!\ell_2!\ldots\ell_k!} e'^{\ell_1}_{\delta,\gamma} e'^{\ell_2}_{2\delta,\gamma} \ldots e'^{\ell_k}_{k\delta,\gamma}.$$

We use the two above equalities to calculate Λ_k for small k. For $k=1$ we obtain

$$C_1 = C^{(1)},$$

where

$$\begin{aligned} C^{(1)} = q^{-(4H_1+2H_2+8)/3} + q^{(2H_1-2H_2-2)/3} + q^{(2H_1+4H_2+4)/3} \\ + \kappa_q^2 F_1 E_1 q^{-(H_1+2H_2+5)/3} + \kappa_q^2 F_2 E_2 q^{(2H_1+H_2+1)/3} \\ + \kappa_q^2 F_3 E_3 q^{-(H_1-H_2-1)/3} - \kappa_q^3 F_3 E_1 E_2 q^{-(H_1-H_2+2)/3}. \end{aligned} \quad (4.65)$$

Note that $C^{(1)}$ belongs to the centre of $\mathrm{U}_q(\mathfrak{sl}_3)$. For $k=2$ we see that

$$C_2 = 2C^{(2)} + C^{(1)2},$$

where

$$\begin{aligned} C^{(2)} = -q^{-(2H_1+4H_2+10)/3} - q^{-(2H_1-2H_2+4)/3} - q^{(4H_1+2H_2+2)/3} \\ - \kappa_q^2 F_1 E_1 q^{(H_1+2H_2-1)/3} - \kappa_q^2 F_2 E_2 q^{-(2H_1+H_2+7)/3} \\ - \kappa_q^2 F_3 E_3 q^{(H_1-H_2-1)/3} - \kappa_q^3 F_1 F_2 E_3 q^{(H_1-H_2-1)/3}. \end{aligned} \quad (4.66)$$

The element $C^{(2)}$ also belongs to the centre of $\mathrm{U}_q(\mathfrak{sl}_3)$. Further calculations give

$$C_3 = 3C^{(3)} + 3C^{(2)}C^{(1)} + C^{(1)3}, \qquad C_4 = 4C^{(3)}C^{(1)} + 2C^{(2)2} + 4C^{(2)}C^{(1)2} + C^{(1)4},$$

where

$$C^{(3)} = q^{-2}. \quad (4.67)$$

It is natural to assume now that C_k are determined by the equality

$$\sum_{k=1}^{\infty} C_k \frac{\zeta^k}{k} = -\log(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta^3). \quad (4.68)$$

Note that in this case $\Lambda(\zeta)$ is uniquely determined by the expansion (4.64) and by the relation

$$\Lambda(q^2\zeta) + \Lambda(\zeta) + \Lambda(q^{-2}\zeta) = -\log(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta^3).$$

Let us show that the above assumption allows to prove the validity of the equalities (4.56) and (4.57).

Note that the definition (3.16) is equivalent to the equality

$$\mathbb{E}'_{\delta,\gamma}(\zeta) = \zeta\big(\mathbb{E}_\gamma(\zeta)\varphi(e_{\delta-\gamma}) - q^{-2}\varphi(e_{\delta-\gamma})\mathbb{E}_\gamma(\zeta)\big).$$

Using (4.58) and the equality

$$\varphi(e_{\delta-\alpha}) = F_1 q^{-(H_1+2H_2)/3},$$

we find that

$$1 + \kappa\mathbb{E}'_{\delta,\alpha}(\zeta) = \overline{\mathbb{M}}'^{-1}_{11}(-q^{-4/3}\zeta)\overline{\mathbb{M}}''_{22}(-q^{-4/3}\zeta).$$

Comparing with (4.50), we see that

$$\mathbb{V}'^{-1}_{11}(\zeta)\mathbb{V}'_{22}(\zeta) = \overline{\mathbb{M}}'^{-1}_{11}(\zeta)\overline{\mathbb{M}}''_{22}(\zeta). \quad (4.69)$$

In a similar way, using (4.59) and the equality

$$\varphi(e_{\delta-\beta}) = -qF_2 q^{(2H_1+H_2)/3} + \kappa_q q^{-1} F_3 E_1 q^{-(H_1-H_2)/3},$$

we conclude that

$$\mathbb{V}'^{-1}_{22}(\zeta)\mathbb{V}'_{33}(\zeta) = \overline{\mathbb{M}}''^{-1}_{22}(\zeta)\big(\overline{\mathbb{M}}''_{33}(\zeta) - \zeta\,\overline{\mathbb{M}}''_{32}(\zeta)\overline{\mathbb{M}}''^{-1}_{22}(\zeta)\overline{\mathbb{M}}''_{23}(\zeta)\big). \tag{4.70}$$

Introduce the notation

$$\mathrm{e}^{\Psi(\zeta)} = \overline{\mathbb{M}}'^{-1}_{11}(\zeta)\mathbb{V}'_{11}(\zeta). \tag{4.71}$$

It follows from (4.69) and (4.70) that

$$\mathrm{e}^{\Psi(\zeta)} = \overline{\mathbb{M}}''^{-1}_{22}(\zeta)\mathbb{V}'_{22}(\zeta) \tag{4.72}$$

and

$$\mathrm{e}^{\Psi(\zeta)} = \big(\overline{\mathbb{M}}''_{33}(q^{-2}\zeta) - q^{-2}\zeta\,\overline{\mathbb{M}}''_{32}(q^{-2}\zeta)\overline{\mathbb{M}}''^{-1}_{22}(q^{-2}\zeta)\overline{\mathbb{M}}''_{23}(q^{-2}\zeta)\big)^{-1}\mathbb{V}'_{33}(\zeta). \tag{4.73}$$

One can demonstrate that

$$(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta^3)^{-1}$$
$$= \overline{\mathbb{M}}'_{11}(q^2\zeta)\overline{\mathbb{M}}''_{22}(\zeta)\big(\overline{\mathbb{M}}''_{33}(q^{-2}\zeta) - q^{-2}\zeta\,\overline{\mathbb{M}}''_{32}(q^{-2}\zeta)\overline{\mathbb{M}}''^{-1}_{22}(q^{-2}\zeta)\overline{\mathbb{M}}''_{23}(q^{-2}\zeta)\big).$$

This relation, together with (4.71)–(4.73) and (4.52), gives

$$\Psi(q^2\zeta) + \Psi(\zeta) + \Psi(q^{-2}\zeta) = -\log(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta).$$

Thus, $\Psi(\zeta) = \Lambda(\zeta)$ and the equalities (4.56) and (4.57) are true.

The matrices $\mathbb{M}^{(m_1,m_2)}(\zeta)$ are obtained from $\mathbb{M}(\zeta)$ by applying to its matrix elements the mapping $\pi^{(m_1,m_2)}$. It follows from (4.65), (4.66) and (4.67) that

$$\pi^{(m_1,m_2)}(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta^3)$$
$$= (1 - q^{-2(2m_1+m_2+4)/3}\zeta)(1 - q^{2(m_1-m_2-1)/3}\zeta)(1 - q^{2(m_1+2m_2+2)/3}\zeta).$$

Then the equality (4.68) gives

$$\pi^{(m_1,m_2)}(C_k) = q^{-2(2m_1+m_2+4)k/3} + q^{2(m_1-m_2-1)k/3} + q^{2(m_1+2m_2+2)k/3},$$

and we come to the relation

$$\pi^{(m_1,m_2)}(\Lambda(\zeta))$$
$$= \lambda_3(q^{-2(2m_1+m_2+4)/3}\zeta) + \lambda_3(q^{2(m_1-m_2-1)/3}\zeta) + \lambda_3(q^{2(m_1+2m_2+2)/3}\zeta).$$

In particular, we have

$$\pi^{(1,0)}(\Lambda(\zeta)) = \lambda_3(q^{-4}\zeta) + \lambda_3(\zeta) + \lambda_3(q^2\zeta).$$

Using this equality, we can check the validity of the relation (4.22).

5. FROM $\mathrm{U}_q(\mathfrak{sl}_3)$ TO $\mathrm{U}_q(\mathfrak{gl}_3)$

We used above the homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ to $\mathrm{U}_q(\mathfrak{sl}_3)$ inspired by the homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ to $\mathrm{U}_q(\mathfrak{gl}_3)$ introduced by Jimbo. In this section we describe the formula for the monodromy operator based on the original Jimbo's homomorphism.

First recall the definition of the quantum group $\mathrm{U}_q(\mathfrak{gl}_3)$. Let $\mathfrak{k}$ be the standard Cartan subalgebra of the Lie algebra $\mathfrak{gl}_3$. The elements $K_i = E_{ii}$, $i = 1, 2, 3$, form a basis of $\mathfrak{k}$. The standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_3$ is a subalgebra of $\mathfrak{k}$. Here one has

$$H_1 = K_1 - K_2, \qquad H_2 = K_2 - K_3. \tag{5.1}$$

The root system of $\mathfrak{gl}_3$ is generated by the simple roots α_i, $i = 1, 2$, given by the relations

$$\alpha_j(K_i) = a_{ij}, \tag{5.2}$$

with

$$(a_{ij}) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Let $\hbar$ again be a complex number and $q = \exp \hbar$. The quantum group $\mathrm{U}_q(\mathfrak{gl}_3)$ is a unital associative $\mathbb{C}$-algebra generated by the elements E_i, F_i, $i = 1, 2$, and q^X, $X \in \mathfrak{k}$, with the relations (2.3)–(2.6), where $\alpha_i \in \mathfrak{k}$ are given by the relation (5.2) and H_i by the equalities (5.1). The analogue of the set of automorphisms (2.8) and (2.9) is

$$E_i \to \nu_i E_i \, q^{\sum_{j=1}^3 \nu_{ij} K_j}, \qquad F_i \to \nu_i^{-1} q^{-\sum_{j=1}^3 \nu_{ij} K_j} F_i, \tag{5.3}$$

$$q^X \to q^X, \tag{5.4}$$

where ν_i are arbitrary nonzero complex numbers and ν_{ij} are complex numbers satisfying the relation

$$\nu_{12} - \nu_{13} = \nu_{21} - \nu_{22}.$$

The homomorphism from $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$ to $\mathrm{U}_q(\mathfrak{gl}_3)$ found by Jimbo is defined by the equalities

$$\begin{aligned}
&\varphi_J(q^{\nu h_0}) = q^{-\nu(K_1 - K_3)}, && \varphi_J(q^{\nu h_1}) = q^{\nu(K_1 - K_2)}, && \varphi_J(q^{\nu h_2}) = q^{\nu(K_2 - K_3)}, \\
&\varphi_J(e_0) = F_3 \, q^{-(K_1 + K_3)}, && \varphi_J(e_1) = E_1, && \varphi_J(e_2) = E_2, \\
&\varphi_J(f_0) = E_3 \, q^{K_1 + K_3}, && \varphi_J(f_1) = F_1, && \varphi_J(f_2) = F_2.
\end{aligned}$$

Comparing with (4.3)–(4.5), we see that

$$\varphi_J(e_0) = q^{-2K/3} \varphi(e_0), \qquad \varphi_J(f_0) = q^{2K/3} \varphi(f_0),$$

where

$$K = K_1 + K_2 + K_3.$$

The action of φ_J on the other generators is the same as the action of φ. It is not difficult to understand that we can obtain the expression for the monodromy operator related to the homomorphism φ_J from the monodromy operator related to the homomorphism φ with the help of the substitution

$$\zeta^{s_\delta} \to q^{2/3} q^{-2K/3} \zeta^{s_\delta}.$$

The result of this substitution is

$$\mathbb{M}(\zeta) = q^{-K/3} \mathrm{e}^{\Lambda(\zeta^{s_\delta})} \begin{pmatrix} q^{K_1} - \zeta^{s_\delta} q^{-K_1} & \zeta^{s_\delta - s_\alpha} \kappa_q q^{-K_1} F_1 & \zeta^{s_\delta - s_{\alpha+\beta}} \kappa_q q^{-K_1} F_3 \\ \zeta^{s_\alpha} \kappa_q E_1 q^{K_1} & q^{K_2} - \zeta^{s_\delta} q^{-K_2} & \zeta^{s_\delta - s_\beta} \kappa_q q^{-K_2} F_2 \\ \zeta^{s_{\alpha+\beta}} \kappa_q E_3 q^{K_1} & \zeta^{s_\beta} \kappa_q E_2 q^{K_2} & q^{K_3} - \zeta^{s_\delta} q^{-K_3} \end{pmatrix}.$$

Here $\Lambda(\zeta)$ has the form (4.64) where C_k determined by the equality (4.68) with

$$C^{(1)} = q^{-2K_1-2} + q^{-2K_2} + q^{-2K_3+2} + \kappa_q^2 F_1 E_1 q^{-K_1-K_2-1} \\ + \kappa_q^2 F_2 E_2 q^{-K_2-K_3+1} + \kappa_q^2 F_3 E_3 q^{-K_1-K_3+1} - \kappa_q^3 F_3 E_1 E_2 q^{-K_1-K_3}, \quad (5.5)$$

$$C^{(2)} = -q^{-2K_1-2K_2-2} - q^{-2K_1-2K_3} - q^{-2K_2-2K_3+2} - \kappa_q^2 F_1 E_1 q^{-K_1-K_2-2K_3+1} \\ - \kappa_q^2 F_2 E_2 q^{-2K_1-K_2-K_3-1} - \kappa_q^2 F_3 E_3 q^{-K_1-2K_2-K_3+1} - \kappa_q^3 F_1 F_2 E_3 q^{-K_1-2K_2-K_3+1}, \quad (5.6)$$

$$C^{(3)} = q^{-2(K_1+K_2+K_3)}. \quad (5.7)$$

Applying to the matrix elements of $\mathbb{M}(\zeta)$ the automorphism (5.3), (5.4) with

$$\nu_1 = q^{-1/2}, \qquad \nu_2 = q^{-1/2},$$

$$\nu_{11} = -1/2, \quad \nu_{12} = 1/2, \quad \nu_{13} = 0, \quad \nu_{21} = 0, \quad \nu_{22} = -1/2, \quad \nu_{23} = 1/2,$$

we obtain

$$\mathbb{M}(\zeta) = q^{-K/3} \mathrm{e}^{\Lambda(\zeta^{s_\delta})} \\ \times \begin{pmatrix} q^{K_1} - \zeta^{s_\delta} q^{-K_1} & \zeta^{s_\delta - s_\alpha} \kappa_q q^{-(K_1+K_2-1)/2} F_1 & \zeta^{s_\delta - s_{\alpha+\beta}} \kappa_q q^{-(K_1+K_3-1)/2} F_3 \\ \zeta^{s_\alpha} \kappa_q E_1 q^{(K_2+K_1-1)/2} & q^{K_2} - \zeta^{s_\delta} q^{-K_2} & \zeta^{s_\delta - s_\beta} \kappa_q q^{-(K_2+K_3-1)/2} F_2 \\ \zeta^{s_{\alpha+\beta}} \kappa_q E_3 q^{(K_3+K_1-1)/2} & \zeta^{s_\beta} \kappa_q E_2 q^{(K_3+K_2-1)/2} & q^{K_3} - \zeta^{s_\delta} q^{-K_3} \end{pmatrix}.$$

This expression is fully consistent with the formula given by Jimbo [16].

The finite dimensional representation $\pi^{(\ell_1,\ell_2,\ell_3)}$ of $\mathrm{U}_q(\mathfrak{gl}_3)$ with the highest weight (ℓ_1, ℓ_2, ℓ_3), $\ell_1, \ell_2, \ell_3 \in \mathbb{N}_0$, and the highest weight vector v_0 is characterised by the equalities

$$q^{\nu K_1} v_0 = q^{\nu \ell_1} v_0, \quad q^{\nu K_2} v_0 = q^{\nu \ell_2} v_0, \quad q^{\nu K_3} v_0 = q^{\nu \ell_3} v_0, \quad E_1 v_0 = 0, \quad E_2 v_0 = 0.$$

Its restriction to the subalgebra $\mathrm{U}_q(\mathfrak{sl}_3)$ is the representation $\pi^{(\ell_1-\ell_2,\ell_2-\ell_3)}$.

Using the relations (5.5)–(5.7), we obtain

$$\pi^{(\ell_1,\ell_2,\ell_3)}(1 - C^{(1)}\zeta - C^{(2)}\zeta^2 - C^{(3)}\zeta^3) = (1 - q^{-2\ell_1-2}\zeta)(1 - q^{-2\ell_2}\zeta)(1 - q^{-2\ell_3+2}\zeta).$$

The equality (4.68) now gives

$$\pi^{(\ell_1,\ell_2,\ell_3)}(C_k) = q^{-2(\ell_1+1)k} + q^{-2\ell_2 k} + q^{-2(\ell_3-1)k}$$

and we come to the relation

$$\pi^{(\ell_1,\ell_2,\ell_3)}(\Lambda(\zeta)) = \lambda_3(q^{-2\ell_1-2}\zeta) + \lambda_3(q^{-2\ell_2}\zeta) + \lambda_3(q^{-2\ell_3+2}\zeta).$$

It is worth to note here that, using the representation $\pi^{(1,0,0)}$, we again obtain the expression for the R-matrix given in section 4.1.

6. Conclusions

Starting with the expression for the universal R-matrix given by Khoroshkin and Tolstoy [23], we constructed the basic monodromy operators for the case of the quantum group $\mathrm{U}_q(\mathcal{L}(\mathfrak{sl}_3))$. We see that despite of the fact that the formula given in [23] is rather formal one can obtain explicit and sensible results. It is important that we have the exact result with the explicit form of the factors belonging to the centre of the quantum groups $\mathrm{U}_q(\mathfrak{sl}_3)$ and $\mathrm{U}_q(\mathfrak{gl}_3)$. An interesting byproduct of our work is the expressions for quantum Casimir elements of $\mathrm{U}_q(\mathfrak{sl}_3)$ and $\mathrm{U}_q(\mathfrak{gl}_3)$, cf. the paper [21].

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